Reliability Capacity of Half-Duplex Channels with Strict Deadlines

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***Abstract*—A fundamental characterization of a half-duplex wireless system with packet losses under traffic with hard deadlines is instrumental to understanding and developing efficient, coding aware policies for real-time applications. We set forth the concept of reliability capacity with a limited number of transmissions and provide closed-form upper and lower bounds for this capacity. We show that the reliability capacity converges to the capacity of a classical erasure channel as the deadline constraint is relaxed. In our framework, the effect of feedback is analyzed both in terms of the reliability capacity and in terms of its advantages towards the mean number of packets that can be transmitted reliably. Optimal schemes for leveraging feedback are presented and the results show that a judicious use of even a single feedback packet can have a significant impact on the mean performance.**

I. Introduction

Timely and reliable transmission of real-time data streams over half-duplex wireless networks is particularly challenging due to the presence of random packet losses and the inherent cost of feedback. In the presence of traffic with hard deadlines, a judicious choice of when to request feedback is instrumental towards achieving these goals.

One of the key questions to ask is how many packets can we send reliably, if we are limited in the number of transmissions. This question has a parallel at the physical layer, where recent work [1], [2] analyzes the maximum coding rate achievable at a given blocklength and error probability. For multiaccess fading channels, [3] investigates the maximum achievable rate with delay independent of how slow the fading is. Our work goes beyond these ideas to incorporate a feedback mode that shares the same channel as the data links. The transmitter still gets a limited number of transmissions, but it can adapt to the events at the receiver with a negotiation cost (channel usage).

Related work on joint design of scheduling and feedback for real-time systems has considered dynamic policies to decide when to send the control messages [4], or how to adapt the rate of control messages [5], but without incorporating the benefits of coding across packets.

The use of feedback is known to enhance network coding performance. Online network coding mechanisms rely on feedback transmitted in parallel channels for maintaining manageable queues [6] and to reducing the decoding delay of individual packets [7]. Recent work on half-duplex network coding mechanisms [8] propose the use of feedback to request additional coded symbols and prove that there exists an optimal number of coded symbols that can be transmitted before the sender receives an acknowledgment.

Aiming to characterize fundamental limits and practical mechanisms that leverage feedback and coding to guarantee packet delivery for traffic with strict deadlines, we make the following contributions:

- *Concept of Reliability Capacity with Limited Transmissions:* We propose a metric for measuring the capacity of a channel given a reliability constraint and maximum number of allowed transmissions. Both upper and lower bounds based on Hoeffding's inequality are provided. These bounds are asymptotically tight.
- *Concept of Minimum Time Required for Reliability:* We set forth a metric that measures the number of time slots necessary to reliably transmit a given number of packets. Asymptotically tight lower and upper bounds are provided.
- *Impact of Feedback:* We illustrate that f feedback packets do not compromise reliability capacity beyond f slots for regions of interest. We illustrate that even a judicious use of a single feedback packet is instrumental towards increasing the mean number of packets delivered to the end receiver.

The remainder of the paper is organized as follows. Section II and Section III discuss the notion of ϵ-reliability capacity and its dual, respectively. In Section IV, we present two coding schemes based on a single feedback transmission, which we analyze in Section V. Finally, Section VI offers some concluding remarks.

II. Reliability Capacity with Limited Time

We are interested in the reliable transmission of n packets over a packet erasure channel, within a limited time interval. We start with a formal definition of the fundamental limit for communication in such setting.

Definition 1: Let T denote the number of available time slots and let e denote the packet erasure probability. Let $\epsilon > 0$. The ϵ-reliability capacity of a packet erasure channel is defined as $C_\epsilon^T = \max\{n : \mathcal{P}(S_T \geq n) \geq 1 - \epsilon\}$, where S_T denotes the number of successes in T Bernoulli trials, with success probability $1 - e$.

The intuition behind *Definition 1* is the following. If the sender wishes to deliver n source packets to the receiver, it requires at least n successful transmissions in the available T time slots. Notice that if the sender employs throughput optimal coding, n successful transmissions are sufficient to provide n source packets to the receiver.

The combinatorial nature of C_ϵ^T does not allow for a closed form expression. We make use of Hoeffding's inequality to obtain bounds on the ϵ-reliability capacity.

Lemma 1 (Hoeffding's inequality [9]): For $X_1, \ldots, X_m$ independent random variables with $\mathcal{P}(X_i \in [a_i, b_i]) = 1, \forall i \in \{1, 2, \ldots, m\}$, if we define $S = X_1 + X_2 + \cdots + X_m$, then

$$\mathcal{P}(S - E(S) \geq m\beta) \leq \exp\left(-2m^2\beta^2 / \sum_{i=1}^{m}(b_i - a_i)^2\right).$$

Using *Lemma 1*, we are now ready to present upper and lower bounds for the ϵ-reliability capacity.

Theorem 1: Let C_ϵ^T denote ϵ-reliability capacity of a memoryless packet erasure channel with erasure probability e. We have that C_ϵ^T verifies, for $0 < \epsilon < 1$,

- $C_\epsilon^T \geq \left\lfloor T(1-e) + 1 - \sqrt{\frac{T}{2}\log\left(\frac{1}{\epsilon}\right)} \right\rfloor$;
- $C_\epsilon^T \leq \left\lceil T(1-e) + \sqrt{\frac{T}{2}\log\left(\frac{1}{1-\epsilon}\right)} \right\rceil$.

Proof: Let $n = \lfloor T(1-e) + 1 - \sqrt{(T/2)\log(1/\epsilon)} \rfloor$. We have that $P(S_T \leq n-1) \leq P(S_T \leq T(1-e) - \sqrt{(T/2)\log(1/\epsilon)})$. Given that we are considering memoryless packet erasure channels, we have that S_T is the sum of T independent Bernoulli variables. From *Lemma 1*, we have that $P(-S_T - E(-S_T) \geq T\beta) \leq \exp(-2T\beta^2)$, which is equivalent to $P(S_T \leq T(1-e) - T\beta) \leq \exp(-2T\beta^2)$. Taking $T\beta = \sqrt{(T/2)\log(1/\epsilon)}$, we have that $P(S_T \leq n-1) \leq \exp\left(-2T(T/2)\log(1/\epsilon)/T^2\right) = \epsilon$, which proves the lower bound.

Now, let $n = \lceil T(1-e) + \sqrt{(T/2)\log(1/(1-\epsilon))} \rceil + 1$. We have that $P(S_T \geq n) < P(S_T \geq n-1)$ and, hence, $P(S_T \geq n) < P(S_T \geq T(1-e) + \sqrt{(T/2)\log(1/(1-\epsilon))})$. From *Lemma 1*, we have that $P(S_T \geq T(1-e) + T\beta) \leq \exp(-2T\beta^2)$. Hence, taking $T\beta = \sqrt{(T/2)\log(1/(1-\epsilon))}$, we have that $P(S_T \geq n) < \exp\left(-2T(T/2)\log(1/(1-\epsilon))/T^2\right) = 1-\epsilon$, which proves the upper bound. ∎

The bounds for C_ϵ^T presented in *Theorem 1* enable the asymptotical analysis of the ϵ-reliability capacity as a function of the number of available time slots, T.

Corollary 1: The ϵ-reliability capacity of a memoryless erasure channel with erasure probability e, C_ϵ^T, verifies, for $0 < \epsilon < 1$, $\lim_{T\to\infty} \frac{C_\epsilon^T}{T} = 1 - e$.

Proof: The result follows from noticing that both the upper and lower bound in *Theorem 1* converge to $1-e$ for asymptotically large T. ∎

Corollary 1 exhibits the connection between the ϵ-reliability capacity and the classical notion of channel capacity, which asserts that the memoryless packet erasure channel has a capacity of $1-e$ packets per transmission. Moreover, *Corollary 1* shows that the bounds in *Theorem 1* are asymptotically tight.

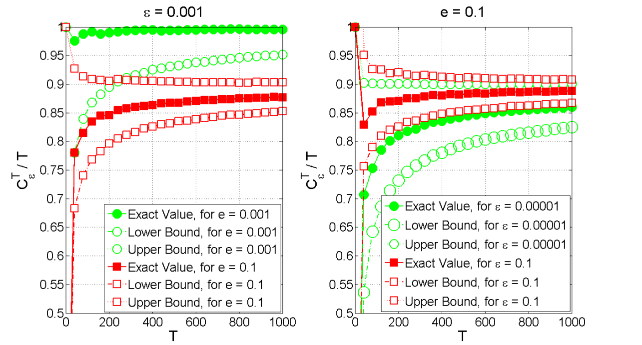

Fig. 1. C_ϵ^T/T as a function of the number of time slots T. On the left, $\epsilon = 0.001$ and $e = \{0.1, 0.001\}$. On the right, $e = 0.1$ and $\epsilon = \{0.00001, 0.1\}$.

In Figure 1, we can observe the ϵ-reliability capacity (divided by the number of available time slots) as a function of T, as well as the upper and lower bounds. As expected, the normalized capacity increases if we allow higher failure probability ϵ. We also observe the convergence of C_ϵ^T/T to $(1-e)$, the classical capacity for the memoryless packet erasure channel.

III. Minimum Time Required for Reliability

We now consider the dual of the ϵ-reliability capacity, i.e. the number of time slots necessary to reliably deliver a given number of packets.

Definition 2: Let n denote the number of packets to transmit, and let e denote the erasure probability. Let $\epsilon > 0$. We define $T_\epsilon^{\min}(n) = \min\{t : \mathcal{P}(S_t \geq n) \geq 1-\epsilon\}$, where S_t denotes the number of successes in t Bernoulli trials, with success probability $1-e$.

$T_\epsilon^{\min}(n)$ represents the minimum number of time slots necessary to ensure the n packets are delivered to the receiver with probability at least ϵ. Similarly to *Definition 1*, the combinatorial nature of $T_\epsilon^{\min}(n)$ does not allow for a closed form expression. Using similar techniques as in the capacity case, we obtain the following bounds for $T_\epsilon^{\min}(n)$.

Theorem 2: For a memoryless packet erasure channel, with erasure probability e, $T_\epsilon^{\min}(n)$ verifies, for $0 < \epsilon < 1$,

- $T_\epsilon^{\min}(n) \leq \frac{\log(1/\epsilon)}{4(1-e)^2} + \frac{n}{1-e} + \alpha(\epsilon, n)$
- $T_\epsilon^{\min}(n) \geq \frac{\log(1/(1-\epsilon))}{4(1-e)^2} + \frac{n-1}{1-e} + 1 - \alpha(1-\epsilon, n-1)$

where $\alpha(x, m) = \frac{1}{4(1-e)^2}\sqrt{\log^2(1/x) + 8(1-e)m\log(1/x)}$.

Proof: *Theorem 1* asserts that, for a given T^*, $C_\epsilon^{T^*} \geq \left\lfloor T^*(1-e) + 1 - \sqrt{(T^*/2)\log(1/\epsilon)} \right\rfloor$. Thus, if $n \leq T^*(1-e) - \sqrt{(T^*/2)\log(1/\epsilon)}$, we have that $P(S_{T^*} \geq n) \geq 1-\epsilon$. Using simple algebraic operations, we have that $n \leq T^*(1-e) - \sqrt{(T^*/2)\log(1/\epsilon)}$ is equivalent to $T^* \geq \frac{\log(1/\epsilon)}{4(1-e)^2} + \frac{n}{1-e} + \alpha(\epsilon, n)$. Therefore, we have that, for all T^* that verifies

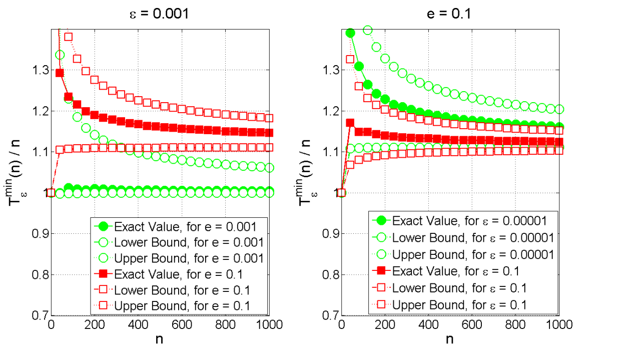

Fig. 2. $T_\epsilon^{\min}(n)/n$ as a function of the number of packets n. On the left, $\epsilon = 0.001$ and $e = \{0.1, 0.001\}$. On the right, $e = 0.1$ and $\epsilon = \{0.00001, 0.1\}$.

the previous inequality, $P(S_{T^*} \geq n) \geq 1 - \epsilon$ and, therefore, the upper bound follows.

From *Theorem 1*, we have that, for a given T^*, $C_\epsilon^{T^*} \leq \left\lceil T^*(1-e) + \sqrt{(T^*/2)\log\left(1/(1-\epsilon)\right)} \right\rceil$. Thus, if $n \geq T^*(1-e) + \sqrt{(T^*/2)\log\left(1/(1-\epsilon)\right)} + 1$, we have $P(S_{T^*} \geq n) < 1-\epsilon$. Through simple algebraic manipulation, we have that $n \geq T^*(1-e) + \sqrt{(T^*/2)\log\left(1/(1-\epsilon)\right)} + 1$ is equivalent to $T^* \leq \frac{\log(1/(1-\epsilon))}{4(1-e)^2} + \frac{n-1}{1-e} - \alpha(1-\epsilon, n-1)$. Hence, we have that, for all T^* that verifies the previous inequality, $P(S_{T^*} \geq n) < 1 - \epsilon$ and, thus, the lower bound follows. ∎

Theorem 2 enables the analysis of the behavior of $T_\epsilon^{\min}(n)$ for asymptotically large n.

Corollary 2: For memoryless packet erasure channels, $T_\epsilon^{\min}(n)$ verifies, for $0 < \epsilon < 1$, $\lim_{n\to\infty} \frac{T_\epsilon^{\min}(n)}{n} = \frac{1}{1-e}$.

Proof: The result follows from noticing that both the upper and lower bound in *Theorem 2* converge to $1/(1-e)$ for asymptotically large n. ∎

In Figure 2, we present the normalized number of time slots necessary to transmit n packets with probability at least $1 - \epsilon$. As expected, increasing the allowed failure probability ϵ allows for transmitting the n packets in a smaller number of time slots. As the number of packet increases, $T_\epsilon^{\min}(n)/n$ decreases, with an asymptotical convergence to $1/(1-e)$.

IV. Feedback to Deliver More Packets

The ϵ-reliability capacity represents the maximum number of packets we can expect to deliver to a receiver through a packet erasure channel, within a limited time interval, for a failure probability of at most ϵ. However, the receiver may get C_ϵ^T successful transmissions before time slot T. For instance, if we consider $e = 0.1$ and compute the probability of decoding C_ϵ^T packets before time slot T, we see that $P(S_T > C_\epsilon^T) \geq 0.99$, for $T \geq 4$ and $\epsilon = \{0.00001, 0.001\}$. For $\epsilon = 0.1$, we have that $P(S_T > C_\epsilon^T) > 0.8$ for $T > 50$.

If the sender is aware that the packets were successfully delivered before the last time slot, it can use the remaining time slots to transmit more packets to the receiver. However, since we are considering half-duplex channels, the sender has to stop transmitting to receive feedback information. This leads to a smaller number of time slots available for transmission and to a potentially smaller ϵ-reliability capacity.

Theorem 3: Consider a packet erasure channel with erasure probability e. The ϵ-reliability capacity in T time slots, C_ϵ^T, verifies $C_\epsilon^T - f \leq C_\epsilon^{T-f} \leq C_\epsilon^T$.

Proof: We start by proving that $C_\epsilon^{T-f} \leq C_\epsilon^T$. We have that $P(S_{T-f} \geq C_\epsilon^{T-f}) = P(S_{T-f} + S_f \geq C_\epsilon^{T-f})$ and thus, since $S_f \leq f$, $P(S_{T-f} \geq C_\epsilon^{T-f}) \leq P(S_{T-f} \geq C_\epsilon^{T-f})$. By definition, we have that $P(S_{T-f} \geq C_\epsilon^{T-f}) \geq 1-\epsilon$. Therefore, $P(S_T \geq C_\epsilon^{T-f}) \geq 1-\epsilon$, which implies that $C_\epsilon^{T-f} \leq C_\epsilon^T$.

Now, notice that $P(S_T \geq C_\epsilon^T) = P(S_{T-f} \geq C_\epsilon^T - S_f)$. Therefore, since $S_f \leq f$, we have that $P(S_T \geq C_\epsilon^T) \leq P(S_{T-f} \geq C_\epsilon^T - f)$. By definition, we have that $P(S_T \geq C_\epsilon^T) \geq 1-\epsilon$. Therefore, $P(S_{T-f} \geq C_\epsilon^T - f) \geq 1-\epsilon$ and, thus, $C_\epsilon^T - f \leq C_\epsilon^{T-f}$. ∎

Theorem 3 asserts that a decrease of f available time slots, for feedback transmissions, does not decrease the capacity by further than f packets.

Let n denote the total number of packets available for transmission, let M denote the number of such packets that must be delivered with probability at least $1-\epsilon$, and let X denote the number of delivered packets after T time slots. If $n \leq C_\epsilon^T$, then $\mathcal{P}(S_T \geq n) \geq 1-\epsilon$ and, hence, $E(X) \geq n(1-\epsilon)$. The space for improvement using feedback transmissions is thus negligible. Moreover, if $M > C_\epsilon^{T-f}$, f feedback transmissions are prohibitive in terms of the reliability demanded. On the other hand, if $n > C_\epsilon^T$ and $M \leq C_\epsilon^{T-f}$, we can use f feedback transmissions to further increase the expected number of delivered packets, without compromising reliability.

A. Coding Schemes

Next, we describe two communication strategies that optimize the expected number of decoded packets, while ensuring the delivery of M packets with probability at least $1-\epsilon$. We focus on the case of a single feedback transmission, i.e. $f = 1$.

First, we set some notation and terminology. We empower the sender with the ability to code across source packets. More precisely, if $p_1, \ldots, p_n$ are the packets to be delivered to the receiver, with $n > C_\epsilon^T$, the sender transmits $c_1, \ldots, c_{T-1}$, with $(c_1, \ldots, c_{T-1}) = (p_1, \ldots, p_n)\mathbf{A}$, where $\mathbf{A}$ is a $n \times (T-1)$ coding matrix. These linear operations are performed over some finite field $\mathbb{F}_q$. We say that a $l \times m$ matrix is *throughput optimal* if any square sub-matrix composed of m rows is invertible.

We start by considering the general case of using throughput optimal coding matrices with no specific structure. Here, the receiver uses one feedback transmission to announce how many degrees of freedom it has received up to that point.

Definition 3 (Coding with Feedback): The source starts by transmitting $c_1, \ldots, c_{T_1}$ in the first T_1 time slots, with $(c_1, \ldots, c_{T-1}) = (p_1, \ldots, p_M)\mathbf{A}_{\text{code-1}}$, where $\mathbf{A}_{\text{code-1}}$ is a $M \times T_1$ throughput optimal coding matrix, for $T_1 \geq M$. At time slot T_1+1, the sender stops transmitting, and the receiver announces how many degrees of freedom it has received in the first T_1 time slots, which is denoted by d_1. Next, the sender computes $n_2(d_1) = \max_{k:\mathcal{P}(X<M)\leq\epsilon} E_k(X|D_1 = d_1)$, where $E_k(X|D_1 = d_1)$ denotes the expected number of decoded packets at the end of the T time slots, if k packets are added to the coding set and throughput optimal coding is employed in the remaining $T - T_1 - 1$ time slots. Finally, the sender transmits $c_{T_1+2}, \ldots, c_T$ in the last $T-T_1-1$ time slots, with $(c_{T_1+2}, \ldots, c_T) = (p_1, \ldots, p_{M+n_2^*})\mathbf{A}_{\text{code-2}}$, where $n_2^* = \min\{n_2(d_1), n-M\}$ and $\mathbf{A}_{\text{code-2}}$ is a $(M+n_2^*)\times(T-T_1-1)$ throughput optimal coding matrix.

Next, we focus on a specific structure for the initial coding matrix. More precisely, we consider a systematic approach to the coding process, as follows.

Definition 4 (Systematic Coding with Feedback): The source starts by transmitting $p_1, \ldots, p_{n_1}$ in the first n_1 time slots, with $n_1 \geq M$. At time slot $n_1 + 1$, the sender stops transmitting, and the receiver announces which packets it has so far received. The number of such packets is denoted by r_1 and assume, without loss of generality, that the received packets were $p_1, \ldots, p_{r_1}$. Next, if $n - r_1 \geq T - n_1 - 1$, the sender transmits uncoded packets in the remaining $T - n_1 - 1$. Otherwise, if $n - r_1 < T - n_1 - 1$, the sender computes $n_2(r_1) = \max_{k:\mathcal{P}(X<M)\leq\epsilon} E_k(X|R_1 = r_1)$, where $E_k(X|R_1 = r_1)$ denotes the expected number of decoded packets at the end of the T time slots, if k packets are added to the coding set and throughput optimal coding is employed in the remaining $T - n_1 - 1$ time slots. Then, the sender transmits $c_{n_1+2}, \ldots, c_T$ in the last $T - n_1 - 1$ time slots, with $(c_{n_1+2}, \ldots, c_T) = (p_{r_1+1}, \ldots, p_{n1+n_2^*})\mathbf{A}_{\text{syst}}$, where $n_2^* = \min\{n_2(r_1), n - n_1\}$ and $\mathbf{A}_{\text{syst}}$ is a $(n1 + n_2^* - r_1) \times (T - n_1 - 1)$ throughput optimal matrix.

V. Impact of a Single Feedback Transmission

We have set two communication schemes, for which we now analyze the performance obtained. Recall that the goal is to transmit the largest possible subset of the $n > C_\epsilon^T$ packets in T time slots, with the requirement that $M \leq C_\epsilon^{T-1}$ or more of such packets are delivered with probability $1 - \epsilon$. For that, the receiver is allowed to use one feedback transmission, that consumes an entire time slot. First, we prove that the aforementioned coding schemes do not compromise reliability.

Proposition 1: Both Coding with Feedback and Systematic Coding with Feedback schemes verify $P(X \geq M) \geq 1 - \epsilon$.

Proof: In the Coding with Feedback scheme, we have that $P(X \geq M) = P(X \geq M|D_1 = M)P(D_1 = M) + P(X \geq M|D_1 < M)P(D_1 < M)$. If $D_1 = M$, M packets were already decoded. Thus, $P(X \geq M) = P(D_1 = M)+P(X \geq M|D_1 < M) * (1-P(D_1 = M))$. Given that the sender adds $n_2(d_1) = \max_{k:\mathcal{P}(X<M)\leq\epsilon} E_k(X|D_1 = d_1)$ to the coding set at slot T_1+, we have that $P(X \geq M|D_1 < M) \geq 1 - \epsilon$. Therefore, we have that $P(X \geq M) \geq P(D_1 = M) + (1 - \epsilon)(1 - P(D_1 = M))$, which is equivalent to $P(X \geq M) \geq 1 - \epsilon \cdot (1 - P(D_1 = M))$. Since $1 - P(D_1 = M) \leq 1$, the result for the Coding with Feedback follows. The proof for the Systematic Coding with Feedback follows analogous arguments. ■

We now devote our attention to the computation of the expected number of delivered packets, $E(T)$. Throughout our analysis, we make the simplifying assumption that all the coding matrices used ($\mathbf{A}_{\text{code-1}}$, $\mathbf{A}_{\text{code-2}}$ and $\mathbf{A}_{\text{syst}}$) do not allow for any intermediate decoding, i.e. the receiver only decodes any source packet when it observes a full rank matrix.

Lemma 2: In the Coding with Feedback scheme, let $P_{d_1}^k(x) = P(X = X|D_1 = d_1)$ when the sender adds k packets to the coding set in the last $T_2 = T - T_1 - 1$ time slots. Then, for $d_1 < M$,

$$P_{d_1}^k(x) = \begin{cases} P(S_{T_2} \geq M + k - d_1) & \text{if } x = M + k \\ P(S_{T_2} < M + k - d_1) & \text{if } x = 0 \text{ and } d_1 < M \\ P(S_{T_2} < k) & \text{if } x = d_1 = M \\ 0 & \text{otherwise} \end{cases}$$

Proof: We are considering coding matrices that do not allow for any intermediate decoding, which means that the receiver must obtain all the degrees of freedom to decode any source packet. Therefore, if $d_1 < M$, the receiver has not decoded any source packet by time slot T_1. Moreover, since k new packets are added to the coding set, in the end of the T time slots, the receiver can only decode $M + k$ packets, for which it obtains the required $M + k - d_1$ degrees of freedom with probability $P(S_{T_2} \geq M + k - d_1)$, or it decodes no packet at all, with probability $P(S_{T_2} < M + k - d_1)$. In case $d_1 = M$, the receiver has decoded M packets by slot T_1 and, therefore, at the end of the T time slots, it can only decode k more packets, which occurs with probability $P(S_{T_2} \geq k)$, or keep M decoded packets, with probability $P(S_{T_2} < k)$. ■

Using *Lemma 2*, we are now able to compute the number of packets the Coding with Feedback scheme adds to the coding set, after receiving feedback information.

Proposition 2: In the Coding with Feedback scheme, we have that $n_2(M) = \arg\max_k k \cdot P(S_{T_2} \geq k)$, and for $d_1 < M$, we have that $n_2(d_1) = \arg\max_{k \leq C_\epsilon^{T_2}+d_1-M} (M+k) \cdot P(S_{T_2} \geq M + k - d_1)$.

Proof: From *Definition 3*, recall that $n_2(d_1) = \max_{k:\mathcal{P}(X<M)\leq\epsilon} E_k(X|D_1 = d_1)$. First, if $d_1 = M$, then we have that $\mathcal{P}(X \geq M) = 1$, $\forall k$. From *Lemma 2*, we have that $E_k(X|D_1 = d_1) = M + k \cdot P(S_{T_2} \geq k)$. Therefore, $n_2(M) = \arg\max_k k \cdot P(S_{T_2} \geq k)$, if $d_1 = M$. For $d_1 < M$, by *Lemma 2*, we have that $X = 0$ or $X = M + k$. Therefore, in order to ensure that $P(X < M) \leq \epsilon$, we must have $P(S_{T_2} \geq M + k - d_1) \geq 1 - \epsilon$, i.e. $M + k - d_1 \leq C_\epsilon^{T_2}$ which is equivalent to $k \leq C_\epsilon^{T_2} + d_1 - M$. In this case, we have $E_k(X|D_1 = d_1) = (M+k) \cdot P(S_{T_2} \geq M+k-d_1)$ and, therefore, $n_2(d_1) = \arg\max_{k \leq C_\epsilon^{T_2}+d_1-M} (M + k) \cdot P(S_{T_2} \geq M + k - d_1)$. ■

With *Lemma 2* and *Proposition 2*, we are now ready to compute the expected number of delivered packets, for the scheme presented in *Definition 3*. First, notice that $P(D_1 = M) = P(S_{T_1} \geq M)$ and, for $d_1 < M$, $P(D_1 = d_1) = P(S_{T_1} = d_1)$. We can then compute the required probability distribution by $P(X = x) = \sum_{d_1=0}^{M} P(X = x|D_1 = d_1)P(D_1 = d_1)$, which we then use to compute $E(X)$.

We now follow a similar strategy to compute the expected number of packets delivered by the Systematic Coding with Feedback scheme.

Lemma 3: In the Systematic Coding with Feedback scheme, let $P^k_{r_1}(x) = P(X = X|R_1 = R_1)$ when the sender adds k packets to the coding set in the last $T_2 = T - n_1 - 1$ time slots. Then, if $n - r_1 \geq T_2$, we have that $P^k_{r_1}(x) = P(S_{T_2} = x - r_1)$ and, if $n - r_1 < T_2$,

$$P^k_{d_1}(x) = \begin{cases} P(S_{T_2} \geq k) & \text{if } x = r_1 + k \\ P(S_{T_2} < k) & \text{if } x = r_1 \\ 0 & \text{otherwise} \end{cases}$$

Proof: First, if $n - r_1 \geq T_2$, the sender transmits T_2 uncoded packets and, thus, given that r_1 packets were already received, $P^k_{r_1}(x) = P(S_{T_2} = x - r_1)$. For the case $n - r_1 < T_2$, recall that we assume that the coding matrix $\mathbf{A}_{\text{syst}}$ does not allow for intermediate decodability. Therefore, we have that $X \in \{r_1, r_1 + k\}$ and the receiver must received at least k transmissions in order to properly decoded the k added packets. Therefore, the results follows. ■

Using *Lemma 3*, we are now able to compute the number of packets over which the System Coding with Feedback scheme performs coding, after receiving feedback information.

Proposition 3: Let S_t denote the number of successes in t Bernoulli trials with success probability $1 - e$, and let $T_2 = T - n_1 - 1$. In the Systematic Coding with Feedback scheme, we have that

$$n_2(r_1) = \begin{cases} \arg \max\limits_{k \leq C_\epsilon^{T_2}} k \cdot P(S_{T_2} \geq k), & \text{if } r_1 < M \\ \arg \max\limits_{k} k \cdot P(S_{T_2} \geq k), & \text{if } r_1 \geq M \end{cases}$$

Proof: From *Definition 3*, recall that $n_2(r_1) = \max\limits_{k:\mathcal{P}(X<M)\leq\epsilon} E_k(X|R_1 = r_1)$. First, if $r_1 \geq M$, then we have that $\mathcal{P}(X \geq M) = 1$, $\forall k$. In this case, from *Lemma 3*, we have that $E_k(X|R_1 = r_1) = r_1 + k \cdot P(S_{T_2} \geq k)$. Therefore, $n_2(M) = \arg\max\limits_k k \cdot P(S_{T_2} \geq k)$. If $r_1 < M$, by *Lemma 3*, we have that $X = r_1$ or $X = r_1 + k$. Therefore, in order to ensure that $P(X < M) \leq \epsilon$, the receiver must decode the k additional packets. Therefore, we must have $P(S_{T_2} \geq k) \geq 1 - \epsilon$, i.e. $k \leq C_\epsilon^{T_2}$. In this case, we have $E_k(X|D_1 = d_1) = r_1 + k \cdot P(S_{T_2} \geq k)$ and, therefore, $n_2(d_1) = \arg\max_{k \leq C_\epsilon^{T_2}} k \cdot P(S_{T_2} \geq k)$. ■

With *Lemma 3* and *Proposition 3*, we are now ready to compute the expected number of delivered packets, for the scheme presented in *Definition 4*. First, notice that $P(R_1 = r_1) = P(S_{n_1} = r_1)$. Next, we can compute the required probability distribution by $P(X = x) = \sum_{r_1=0}^{n_1} P(X = x|R_1 = r_1)P(R_1 = r_1)$, which we then use to compute $E(X)$.

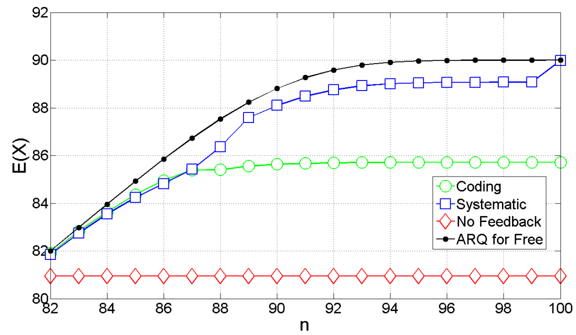

Fig. 3. The expected number of decoded packets, $E(X)$, for $n > C_\epsilon^T$ and for the optimal parameter choice in each scheme. Here, $T = 100$,$e = 0.1$, $\epsilon = 0.001$ and $M = \min\{n, C_\epsilon^{T-1}\}$.

Before presenting the expected number of delivered packets achieved by the two proposed schemes, we first describe an upper bound for such performance metric. If we remove the half-duplex constraint of the channel and we allow the receiver to acknowledge the reception of each individual transmission, the optimal solution is to use Automatic Repeat Request (ARQ), where the sender transmits each packet until receiving a positive acknowledgment. This scheme naturally sets an upper bound on the expected number of delivered packets, given that every successful transmission provides a new packet to the receiver and, since there are no half-duplex constrains, there are T available time slots to transmit. The expected number of delivered packets achieved by the ARQ scheme can be easily computed using a Markov process, where the state represents the number of delivered packets.

As a benchmark, we compare the proposed schemes to the performance obtained when no feedback is available. The expected number of delivered packets is given by $k \cdot P(S_T \geq k)$, where k denotes the number of packets in the coding set. Hence, we compare the aforementioned schemes to the optimal choice of k subject to the reliability constrain, i.e. we compare against $E_{\text{NoFeed}}(X) = \max\limits_{k \leq C_\epsilon^T} k \cdot P(S_T \geq k)$.

In Figure 3, we present the expected number of delivered packets for the optimal choice of parameters in each of the proposed schemes. More precisely, we present $E_{\text{Code}}(X) = \max\limits_{T_1 \geq M} E(X)$ for the Coding with Feedback scheme, and $E_{\text{Code}}(X) = \max\limits_{n_1 \geq M} E(X)$ for the Systematic Coding with Feedback scheme, as well as $E_{\text{ARQ}}(X)$ and $E_{\text{NoFeed}}(X)$. The results show the significant impact that the proper use of a single feedback transmission has on the number of delivered packets, without compromising reliability. The gain is particularly evident when we employ a systematic coding technique, where the expected number of delivered packets goes from 81, in the no feedback case, to 89 in the Systematic Coding with Feedback case, roughly a 10% increase, and only one packet away from the optimal ARQ mechanism.

VI. Conclusions

We presented fundamental limits for the transmission of packets with strict deadlines in half-duplex channels using network coding. In particular, the concept of reliability capacity given a number of available slots before the deadline was introduced and asymptotically tight lower and upper bounds were presented. Our results showed that the use of feedback causes some degradation on the reliability capacity, although bounded by the number of slots dedicated to feedback. We then proposed optimal mechanisms to leverage feedback in order to improve the mean delivery of packets. We provided numerical results that show that even if the reliability capacity is somewhat degraded with the use of feedback, the mean number of delivered packets can be significantly improved. Future work shall extend these results to the case of multiple feedback transmissions, as well as including multiple receivers.

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